On Cartan Spaces with m-th Root Metrics

A. Tayebi, A. Nankali and E. Peyghan

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Abstract

In this paper, we define some non-Riemannian curvature properties for Cartan spaces. We consider Cartan space with the m-th root metric. We prove that every m-th root Cartan space of isotropic Landsberg curvature, or isotropic mean Landsberg curvature, or isotropic mean Berwald curvature reduces to a Landsberg, weakly Landsberg and weakly Berwald space, respectively. Then we show that m-th root Cartan space of almost vanishing $\mathbf{H}$-curvature satisfies $\mathbf{H} = 0$.

Keywords: Landsberg curvature, mean Landsberg curvature, mean Berwald curvature, $\mathbf{H}$-curvature.[1]

1 Introduction

É. Cartan has originally introduced a Cartan space, which is considered as dual of Finsler space [4]. Then Rund and Brickell studied the relation between these two spaces [3][16]. The theory of Hamilton spaces was introduced by Miron [11]. He proved that Cartan space is a particular case of Hamilton space.

Let us denote the Hamiltonian structure on a manifold M by $(M, H(x,p))$. If the fundamental function $H(x,p)$ is 2-homogeneous on the fibres of the cotangent bundle T^*M, then the notion of Cartan space is obtained [10][14][15]. Indeed, the modern formulation of the notion of Cartan spaces is due of the Miron [11][12]. Based on the studies of Kawaguchi [6], Miron [10], Hrimiuc-Shimada [5], Anastasiei-Antonelli [2], Mazėtis [7][8][9], Urbonas [22] etc., the geometry of Cartan spaces is today an important chapter of differential geometry.

Under Legendre transformation, the Cartan spaces appear as the dual of the Finsler spaces [11]. Finsler geometry was developed since 1918 by Finsler, Cartan, Berwald, Akbar-Zadeh, Matsumoto, Shen and many others, see [1][17]. Using this duality several important results in the Cartan spaces can be obtained: the canonical nonlinear connection, the canonical metrical connection, the notion of (α, β)-metrics, the theory of m-root metrics, etc [7][8][13]. Therefore, the theory of Cartan spaces has the same symmetry and beauty like Finsler geometry. Moreover, it gives a geometrical framework for the Hamiltonian theory of Mechanics or Physical fields.

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The theory of m-th root metric has been developed by H. Shimada [18], and applied to Biology as an ecological metric. It is regarded as a direct generalization of Riemannian metric in a sense, i.e., the second root metric is a Riemannian metric. Recently studies, shows that the theory of m-th root Finsler metrics play a very important role in physics, theory of space-time structure, general relativity and seismic ray theory [19][20][21].

An n-dimensional Cartan space C^n with m-th root metric is a Cartan structure $C^n = (M^n, K(x,p))$ on differentiable n-manifold M^n equipped with the fundamental function $K(x,p) = \sqrt[m]{a^{i_1 i_2 \cdots i_m}(x) p_{i_1} p_{i_2} ... p_{i_m}}$ where $a^{i_1 i_2 \cdots i_m}(x)$, depending on the position alone, is symmetric in all the indices $i_1, i_2, ..., i_m$ and $m \geq 3$. The Hessian of K^2 give us the fundamental tensor $\mathbf{g}$ of Cartan space. Taking a vertical derivation of $\mathbf{g}$ give us the Cartan torsion $\mathbf{C}$. The rate of change of the Cartan torsion along geodesics, $\mathbf{L}$ is said to be Landsberg curvature. A Cartan metric with $\mathbf{L} = cF\mathbf{C}$ is called isotropic Landsberg metric, where $c = c(x)$ is a scalar function on M. In this paper, we prove that every m-root Cartan spaces with isotropic Landsberg curvature is a Landsberg space.

Theorem 1.1. *Let (M, K) be an m-th root Cartan space. Suppose that K is isotropic Landsberg metric, $\mathbf{L} + cK\mathbf{C} = 0$ for some scalar function $c = c(x)$ on M. Then K reduces to a Landsberg metric.*

Taking a trace of Cartan torsion $\mathbf{C}_y$ and Landsberg curvature $\mathbf{L}_y$ give us the mean Cartan torsion $\mathbf{I}_y$ and mean Landsberg curvature $\mathbf{J}_y$, respectively. A Cartan metric with $\mathbf{J} = 0$ and $\mathbf{J} = cF\mathbf{I}$ is called weakly Landsberg and isotropic mean Landsberg metric, respectively, where $c = c(x)$ is a scalar function on M. We show that every m-root Cartan spaces of isotropic mean Landsberg curvature reduces to weakly Landsberg space.

Theorem 1.2. *Let (M, K) be an m-th root Cartan space. Suppose that K has isotropic mean Landsberg curvature, $\mathbf{J} + cK\mathbf{I} = 0$ for some scalar function $c = c(x)$ on M. Then K reduces to a weakly Landsberg metric.*

Taking a trace of Berwald curvature of Cartan metric K gives rise the E-curvature. The Cartan metric K with $\mathbf{E} = 0$ and $\mathbf{E} = \frac{n+1}{2}cK\mathbf{h}$ is called weakly Berwald and isotropic mean Berwald metric, respectively, where $c = c(x)$ is a scalar function on M and $\mathbf{h} = h^{ij}dx_i dx_j$ is the angular metric.

Theorem 1.3. *Let (M, K) be an m-th root Cartan space. Suppose that K has isotropic mean Berwald curvature $\mathbf{E} = \frac{n+1}{2}cK\mathbf{h}$, for some scalar function $c = c(x)$ on M. Then K reduces to a weakly Berwald metric.*

Akbar-Zadeh introduces the non-Riemannian quantity $\mathbf{H}$ which is obtained from the mean Berwald curvature bye the covariant horizontal differentiation along geodesics. More precisely, the non-Riemannian quantity $\mathbf{H} = H^{ij} dx_i \otimes dx_j$ is defined by $H^{ij} := E^{ij|s} p_s$. The Cartan metric K is called of almost vanishing $\mathbf{H}$-curvature if $H^{ij} = \frac{n+1}{2K}\theta h^{ij}$, where θ is a 1-form on M.

Theorem 1.4. *Let (M, K) be an n-dimensional m-th root Cartan space. Suppose that K has almost vanishing $\mathbf{H}$-curvature, $\mathbf{H} = \frac{n+1}{2}K^{-1}\theta\mathbf{h}$ for some 1-form θ on M. Then $\mathbf{H} = \mathbf{0}$.*

2 Preliminaries

A Cartan spaces is a pair $C^n = (M^n, K(x,p))$ such that the following axioms hold good:
1. K is a real positive function on the cotangent bundle T^*M, differentiable on $T^*M_0 := T^*M \setminus \{0\}$ and continuous on the null section of the canonical projection

$$\pi^* : T^*M \to M;$$

2. K is positively 1-homogenous with respect to the momenta p_i;
3. The Hessian of K^2, with the elements

$$g^{ij}(x,p) = \frac{1}{2}\frac{\partial^2 K^2}{\partial p_i \partial p_j}$$

is positive-defined on T^*M_0.

An n-dimensional Cartan space C^n with m-th root metric is by definition a Cartan structure $C^n = (M^n, K(x,p))$ on differentiable n-manifold M^n equipped with the fundamental function $K(x,p)$ such that

$$K(x,p) = \sqrt[m]{a^{i_1 i_2 \dots i_m}(x) p_{i_1} p_{i_2} \dots p_{i_m}}$$

where $a^{i_1 i_2 \dots i_m}(x)$, depending on the position alone, is symmetric in all the indices $i_1, i_2, ..., i_m$ and $m \geq 3$.

From $K(x,y)$ we define Cartan symmetric tensors of order r $(1 \leq r \leq m-1)$ with the components

$$a^{i_1 i_2 \dots i_r}(x,p) = \frac{1}{K^{m-r}} a^{i_1 i_2 \dots i_r j_1 j_2 \dots j_{m-r}} p_{j_1} p_{j_2} \dots p_{j_{m-r}}$$

Thus we have

$$\begin{aligned} a^i &= [a^{i i_2 i_3 \dots i_m}(x) p_{i_2} p_{i_3} \dots p_{i_m}] \big/ K^{m-1}, \\ a^{ij} &= [a^{ij i_3 i_4 \dots i_m}(x) p_{i_3} p_{i_4} \dots p_{i_m}] \big/ K^{m-2}, \\ a^{ijk} &= [a^{ijk i_4 i_5 \dots i_m}(x) p_{i_4} p_{i_5} \dots p_{i_m}] \big/ K^{m-1}. \end{aligned}$$

The normalized supporting element is given by

$$l^i = \dot{\partial}^i K,$$

where $\dot{\partial}^i = \frac{\partial}{\partial p_i}$. The fundamental metrical d-tensor is

$$g^{ij} = \frac{1}{2} \dot{\partial}^i \dot{\partial}^j K^2$$

and the angular metrical d-tensor is given by

$$h^{ij} = K \dot{\partial}^i \dot{\partial}^j K.$$

The following hold

$$l^i = a^i,$$
$$g^{ij} = (m-1)a^{ij} - (m-2)a^i a^j,$$
$$h^{ij} = (m-1)(a^{ij} - a^i a^j).$$

From the positively 1-homogeneity of the m-th root Cartan metrical function, it follows that

$$K^2(x,p) = g^{ij}(x,p)p_i p_j = a^{ij}(x,p)p_i p_j.$$

Since $det(g^{ij}) = (m-1)^{n-1} det(a^{ij})$, the regularity of the m-th metric is equivalent to $det(a^{ij}) \neq 0$. Let us suppose now that the d-tensor a^{ij} is regular, that is there exists the inverse matrix $(a^{ij})^{-1} = (a_{ij})$. Obviously, we have

$$a_i.a^i = 1,$$

where

$$a_i = a_{is}a^s = \frac{p_i}{K}.$$

Under these assumptions, we obtain the inverse components $g_{ij}(x,p)$ of the fundamental metrical d-tensor $g^{ij}(x,p)$, which are given by

$$g_{ij} = \frac{1}{m-1}a_{ij} + \frac{m-2}{m-1}a_i a_j. \tag{1}$$

We have

$$\dot{\partial}^k(a^{ij}) = \frac{(m-2)}{K}[a^{ijk} - a^{ij}a^k],$$
$$\dot{\partial}^k(a^i) = \frac{(m-1)}{K}[a^{ik} - a^i a^k],$$
$$\dot{\partial}^k(a^i a^j) = \frac{(m-1)}{K}[a^{ik}a^j + a^{jk}a^i - 2a^i a^j a^k].$$

The Cartan tensor $C^{ijk} = -\frac{1}{2}(\dot{\partial}^k g^{ij})$ are given in the form

$$C^{ijk} = -\frac{(m-1)(m-2)}{2K}(a^{ijk} - a^{ij}a^k - a^{jk}a^i - a^{ki}a^j + 2a^i a^j a^k). \tag{2}$$

The m-th Christoffel symbols is defined by

$$\{i_1 ... i_m, j\} = \frac{1}{2(m-1)}(\partial^{i_1} a^{i_2 \cdots i_m j} + \partial^{i_2} a^{i_3 \cdots i_m i_1 j} + \cdots + \partial^{i_m} a^{i_1 \cdots i_{m-1} j} - \partial^j a^{i_1 \cdots i_m}), \tag{3}$$

where the cyclic permutation is applied to $(i_1 ... i_m)$ in the first m terms of the right-hand side.

Now, if we write the equations of geodesics in the usual form

$$\frac{d^2 x_i}{ds^2} + 2G_i(x, \frac{dx}{ds}) = 0, \tag{4}$$

then the quantities $G_i(x,y)$ are given by

$$a^{hr}G_r = \frac{1}{mK^{m-2}}\{00...0,h\}, \tag{5}$$

where we denote by the index 0 the multiplying by p_i as usual, that is

$$\{00...0,h\} = \{i_1i_2...i_m,h\}p_{i_1}p_{i_2}...p_{i_m}, a^{hr} = a^{hr00...0}/K^{m-2}.$$

Using the definition of a^{hr}, we can write (5) in the form

$$a^{hr00...0}G_r = \frac{1}{m}\{00...0,h\}. \tag{6}$$

Differentiating of (6) with respect to p_i yields

$$a^{hr00...0}G^i_r + (m-2)a^{hri00...0}G^r = \{i00...0,h\}, \tag{7}$$

where $G^i_r = \dot{\partial}^i G_r$. By differentiating of (7) with respect to p_j, we have

$$\begin{aligned} a^{hr00...0}G^{ij}_r + (m-2)\big[a^{hrj00...0}G^i_r + a^{hri00...0}G^j_r\big] + (m-2)(m-3)a^{hrij00...0}G_r \\ = (m-1)\{ij00...0,h\}, \end{aligned}$$

where $G^{ij}_r = \dot{\partial}^j G^i_r$ constitute the coefficients of the Berwald connection $B\Gamma = (G^{ij}_r, G^i_r)$. The above equations can be written in the following forms

$$K^{m-3}\big[Ka^{hr}G^i_r + (m-2)a^{hri}G_r\big] = \{i00...0,h\}, \tag{8}$$

and

$$\begin{aligned} K^{m-3}\Big[Ka^{hr}G^{ij}_r + (m-2)(a^{hrj}G^i_r + a^{hri}G^j_r)\Big] + K^{m-4}(m-2)(m-3)a^{hrij}G_r \\ = (m-1)\{ij00...0,h\}. \end{aligned} \tag{9}$$

By differentiation of (9) with respect to p_k, we get the Berwald curvature of Berwald connection as follows

$$\begin{aligned} K^{m-3}\Big[\, & Ka^{hr}G^{ijk}_r + (m-2)[a^{hir}G^{jk}_r + (i,j,k)]\Big] \\ & + (m-2)(m-3)K^{m-5}\Big[K[a^{hijr}G^k_r + (i,j,k)] + (m-4)a^{hijkr}G_r\Big] \\ & = (m-1)(m-2)\{ijk00...0,h\}, \end{aligned} \tag{10}$$

where $\{...,(ijk)\}$ shows the cyclic permutation of the indices i,j,k and summation. Multiplying (10) with p_h yields

$$\begin{aligned} K^{m-2}\Big[\, & p^rG^{ijk}_r + (m-2)[a^{ir}G^{jk}_r + (i,j,k)]\Big] \\ & + (m-2)(m-3)K^{m-4}\Big[K[a^{ijr}G^k_r + (i,j,k)] + (m-4)a^{ijkr}G_r\Big] \\ & = (m-1)(m-2)\{ijk00...0,0\}. \end{aligned} \tag{11}$$

Remark 2.1. In the equations (10) and (11), we have some terms with coefficients $(m-3)$ and $(m-4)$. We shall be concerned mainly with cubic metric $(m=3)$ and quartic metric $(m=4)$

$$K^3 = a^{ijk}(x)p_i p_j p_k \quad , \quad K^4 = a^{hijk}(x)p_h p_i p_j p_k.$$

For these metrics, it is supposed that the terms with $(m-3)$ and $(m-4)$ vanish, respectively. For instance, (10) of a cubic metric is reduced to following

$$K a^{hr} G_r^{ijk} + \{a^{hir} G_r^{jk} + (i,j,k)\} = \{ijk, h\}.$$

3 Proof of Theorem 1.1

In this section, we are going to prove Theorem 1.1. First, we remark the following.

Lemma 3.1. ([**?**]) Let (M, K) be an m-th root Cartan space. Then the spray coefficients of K are given by following

$$G_r = \frac{1}{m}\{00...0, h\} \diagup a^{hr00...0}.$$

Now, we can prove the Theorem 1.1.

Proof of Theorem 1.1: By assumption, the Cartan metric K has isotropic Landsberg curvature $\mathbf{L} = cK\mathbf{C}$ where $c = c(x)$ is a scalar function on M. By definition, we have

$$L^{ijk} = -\frac{1}{2} p^s G_s^{ijk},$$

where

$$\begin{aligned} p^s = g^{sj} p_j &= [(m-1)a^{sj} - (m-2)a^s a^j] p_j \\ &= (m-1)K a^s - (m-2)K a^s \\ &= K a^s \\ &= a^{s00...0} \diagup K^{m-2}. \end{aligned} \tag{12}$$

Then we get

$$L^{ijk} = -\frac{1}{2} a^{s00...0} \diagup K^{m-2} G_s^{ijk}.$$

By assumption, we have

$$-\frac{1}{2K^{m-2}} a^{s00...0} G_s^{ijk} = cKC^{ijk}$$

or equivalently

$$a^{s00...0} G_s^{ijk} = \frac{c}{K^{2-m}}(m-1)(m-2)(a^{ijk} - a^{ij}a^k - a^{jk}a^i - a^{ki}a^j + 2a^i a^j a^k). \tag{13}$$

By Lemma 3.1, the left-hand side of (13) is rational function, while its right-hand side is an irrational function. Thus, either $c = 0$ or a satisfies the following

$$a^{ijk} - a^{ij}a^k - a^{jk}a^i - a^{ki}a^j + 2a^ia^ja^k = 0. \tag{14}$$

Plugging (17) into (2) implies that $C^{ijk} = 0$. Hence, K is Riemannian metric, which contradicts with our assumption. Therefore, $c = 0$. This completes the proof. □

4 Proof of Theorem 1.2

The quotient $\mathbf{J}/\mathbf{I}$ is regarded as the relative rate of change of mean Cartan torsion $\mathbf{I}$ along Cartan geodesics. Then K is said to be isotropic mean Landsberg metric if $\mathbf{J} = cK\mathbf{I}$, where $c = c(x)$ is a scalar function on M. In this section, we are going to prove the Theorem 1.2. More precisely, we show that every m-th root isotropic mean Landsberg metric reduces to a weakly Landsberg metric.

Proof of Theorem 1.2: The mean Cartan tensor of K is given by following

$$\begin{aligned} I^i &= g_{jk}C^{ijk} \\ &= \frac{-(m-2)}{2K}\{a^{ij}_j - \delta^i_k a^k - na^i - \delta^i_j a^j + 2a^i\} \\ &= \frac{-(m-2)}{2K}\{a^{ir}_r - na^i\}. \end{aligned}$$

The mean Landsberg curvature of K is given by

$$\begin{aligned} J^i &= g_{jk}L^{ijk} \\ &= [\frac{1}{m-1}a_{jk} + \frac{m-2}{m-1}a_ja_k][-\frac{1}{2}a^{s00\dots0}\diagup K^{m-2}G^{ijk}_s]. \end{aligned}$$

Since $\mathbf{J} = cF\mathbf{I}$, then we have

$$c(m-1)(m-2)\{a^{ijk} - a^{ij}a^k - a^{jk}a^i - a^{ki}a^j + 2a^ia^ja^k\} = a^{s00\dots0}\diagup K^{m-2}G^{ijk}_s.$$

Thus we get

$$a^{s00\dots0}G^{ijk}_s = cK^{m-2}(m-1)(m-2)\{a^{ijk} - a^{ij}a^k - a^{jk}a^i - a^{ki}a^j + 2a^ia^ja^k\}. \tag{15}$$

By Lemma 3.1, the left hand side of (15) is a rational function with respect to y, while its right-hand side is an irrational function with respect to y. Thus, either $c = 0$ or a satisfies the following

$$a^{ijk} - a^{ij}a^k - a^{jk}a^i - a^{ki}a^j + 2a^ia^ja^k = 0.$$

That implies that $C^{ijk} = 0$. Hence, K is Riemannian metric, which contradicts with our assumption. Therefore, $c = 0$. This completes the proof. □

5 Proof of Theorem 1.3

Let (M, K) be a Cartan space of dimension n. Denote by $\tau(x, y)$ the distortion of the Minkowski norm K_x on $T_x^*M_0$, and $\sigma(t)$ be the geodesic with $\sigma(0) = x$ and $\dot{\sigma}(0) = y$. The rate of change of $\tau(x, y)$ along Cartan geodesics $\sigma(t)$ called S-curvature. K is said to have isotropic S-curvature if

$$\mathbf{S} = (n+1)cK.$$

where $c = c(x)$ is a scalar function on M. K is called of almost isotropic S-curvature if

$$\mathbf{S} = (n+1)cK + \eta,$$

where $c = c(x)$ is a scalar function and $\eta = \eta^i(x)p_i$ is a 1-form on M.

Remark 5.1. By taking twice vertical covariant derivatives of the S-curvature, we get the E-curvature

$$E^{ij}(x,p) := \frac{1}{2}\frac{\partial^2 \mathbf{S}}{\partial p_i \partial p_j}.$$

It is remarkable that, we can get the E-curvature by taking a trace of Berwald curvature of Cartan metric K, also. The Cartan metric K is called weakly Berwald metric if $\mathbf{E} = 0$ and is said to have isotropic mean Berwald curvature if $\mathbf{E} = \frac{n+1}{2}cK\mathbf{h}$, where $c = c(x)$ is a scalar function on M and $\mathbf{h} = h^{ij}dx_i dx_j$ is the angular metric.

In this section, we are going to prove an extension of Theorem 1.3. More precisely, we prove the following.

Theorem 5.2. *Let (M, K) be an m-th root Cartan space. Then the following are equivalent:*

a) *K has isotropic mean Berwald curvature, i.e., $\mathbf{E} = \frac{n+1}{2}cK\mathbf{h}$;*

b) *K has vanishing E-curvature, i.e., $\mathbf{E} = 0$;*

c) *K has almost isotropic S-curvature, i.e., $\mathbf{S} = (n+1)cK + \eta$;*

where $c = c(x)$ is a scalar function and $\eta = \eta_i(x)y^i$ is a 1-form on M.

To prove the Theorem 5.2, first we show the following.

Lemma 5.3. *Let (M, K) be an n-dimensional m-th root Cartan space. Then the following are equivalent:*

a) $\mathbf{S} = (n+1)cK + \eta$;

b) $\mathbf{S} = \eta$;

where $c = c(x)$ is a scalar function and $\eta = \eta^i(x)p_i$ is a 1-form on M.

Proof. By lemma 3.1, the E-curvature of an m-th root metric is a rational function . On the other hand, by taking twice vertical covariant derivatives of the **S**-curvature,we get the E-curvature. Thus **S**-curvature is a rational function. Suppose that K has almost isotropic **S**-curvature, $\mathbf{S} = (n+1)cK + \eta$, where $c = c(x)$ is a scalar function and $\eta = \eta^i(x)p_i$ is a 1-form on M. Then the left hand side of $\mathbf{S} - \eta = (n+1)c(x)K$ is rational function while the right hand is irrational function. Thus $c = 0$ and $\mathbf{S} = \eta$. □

Lemma 5.4. *Let (M, K) be an n-dimensional m-th root Cartan space. Then the following are equivalent:*

a) $\mathbf{E} = \frac{n+1}{2}cK\mathbf{h}$;

b) $\mathbf{E} = 0$;

where $c = c(x)$ is a scalar function.

Proof. Suppose that K has isotropic mean Berwald curvature

$$\mathbf{E} = \frac{n+1}{2}cK\mathbf{h}, \tag{16}$$

where $c = c(x)$ is a scalar function. The left hand side of (16), is a rational function while the right hand is irrational function. Thus $c = 0$ and $E = 0$. □

Proof of Theorem 5.2: By Lemmas 5.3 and 5.4, we get the proof.

Corollary 5.5. *Let (M, K) be an n-dimensional m-th root Cartan space. Suppose that K has isotropic* **S**-*curvature,* $\mathbf{S} = (n+1)c(x)K$, *for some scalar function $c = c(x)$ on M. Then* $\mathbf{S} = 0$.

6 Proof of Theorem 1.4

Proof of Theorem 1.4: Let (M, K) be an n-dimensional m-th root Cartan space. Suppose that K be of almost vanishing **H**-curvature, i.e.,

$$H^{ij} = \frac{n+1}{2K}\theta h^{ij}, \tag{17}$$

where θ is a 1-form on M. The angular metric $h^{ij} = g^{ij} - K^2p^ip^j$ is given bye the following

$$h^{ij} = (m-1)(a^{ij} - a^i a^j), \tag{18}$$

Plugging (18) into (17) yields

$$H^{ij} = \frac{n+1}{2K}\theta[(m-1)(a^{ij} - a^i a^j)]. \tag{19}$$

By Lemma 3.1 and $H^{ij} = E^{ij|s}p_s$, it is easy to see that H^{ij} is rational with respect to y. Thus, (19) implies that $\theta = 0$ or

$$(m-1)(a^{ij} - a^i a^j) = 0. \tag{20}$$

By (18) and (20), we conclude that $h^{ij} = 0$, which is impossible. Hence $\theta = 0$ and then $H^{ij} = 0$. $\square$

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Akbar Tayebi and Ali Nankali
Department of Mathematics, Faculty of Science
University of Qom
Qom. Iran
Email: akbar.tayebi@gmail.com
Email: ali.nankali2327@yahoo.com

Esmaeil Peyghan
Department of Mathematics, Faculty of Science
Arak University
Arak 38156-8-8349, Iran
Email: epeyghan@gmail.com